Self-Similar Solutions with Elliptic Symmetry for the Density-Dependent Navier-Stokes Equations in R^N

Manwai Yuen*
Department of Applied Mathematics,
The Hong Kong Polytechnic University,
Hung Hom, Kowloon, Hong Kong

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Abstract

Based on Yuen's solutions with radially symmetry of the pressureless density-dependent Navier-Stokes in R^N, the corresponding ones with elliptic symmetry are constructed by the separation method. In detail, we successfully reduce the pressureless Navier-Stokes equations with density-dependent viscosity into $1+N$ differential functional equations. In particular for $\kappa_1 > 0$ and $\kappa_2 = 0$, the velocity is built by the new Emden dynamical system with force-force interaction:

$$\begin{cases} \ddot{a}_i(t) = \dfrac{-\xi\left(\sum_{k=1}^N \frac{\dot{a}_k(t)}{a_k(t)}\right)}{a_i(t)\left(\prod\limits_{k=1}^N a_k(t)\right)^{\theta-1}} \text{ for } i = 1, 2, ..., N \\ a_i(0) = a_{i0} > 0, \ \dot{a}_i(0) = a_{i1} \end{cases} \tag{1}$$

with arbitrary constants ξ, a_{i0} and a_{i1}. We can show some blowup phenomena or global existences for the obtained solutions. Based on the complication of the deduced Emden dynamical systems, the author conjectures there exist limit cycles or chaos for this kind of flows. Numerical simulation or mathematical proofs for the Emden dynamical systems are expected in the future.

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1 Introduction

The pressureless density-dependent Navier-Stokes equations are

$$\begin{cases} \rho_t + \nabla\cdot(\rho\vec{u}) = 0 \\ \rho\left[\frac{\partial}{\partial t}u_i + \vec{u}\cdot\nabla u_i\right] = \kappa_1\frac{\partial}{\partial x_i}\left(\rho^\theta\nabla\cdot\vec{u}\right) + \kappa_2\nabla\cdot\left(\rho^\theta\nabla u_i\right) \text{ for } i = 1, 2, ..., N \end{cases} \tag{2}$$

where the density $\rho = \rho(t, \vec{x})$ and velocity $\vec{u} = \vec{u}(t, \vec{x}) = (u_1, u_2, ..., u_N) \in R^N$ with $\vec{x} = (x_1, x_2, ..., x_N) \in R^N$ with constants θ, κ_1, $\kappa_2 \geq 0$. When $\theta = 0$, the system is the pressureless Navier-Stokes equation without density-dependent viscosity:

$$\begin{cases} \rho_t + \nabla\cdot(\rho\vec{u}) = 0 \\ \rho\left[\frac{\partial}{\partial t}u_i + \vec{u}\cdot\nabla u_i\right] = \kappa_1\frac{\partial}{\partial x_i}\left(\nabla\cdot\vec{u}\right) + \kappa_2\Delta u_i \text{ for } i = 1, 2, ..., N \end{cases} \tag{3}$$

*E-mail address: nevetsyuen@hotmail.com

In fluid dynamics, the Navier-Stokes equations (2) are the very important mathematical models [4] and [5].

There are intensive papers concerning with the mathematical existence of the Navier-Stokes equations with density-dependent viscosity in the recent literature [1], [2], [3], [6], [7] and [15].

In mathematical physics, constructing particular solutions for the Navier-Stokes equations (2) are an important area for explaining its nonlinear appearance.

In 2008, Yuen first constructed a class of self-similar solutions by the separation method, for the pressureless density-dependent Navier-Stokes equations with radial symmetry [11]. And the separation method is also applied to the other related systems with density-dependent viscosity and radial symmetry [12], [8] and [9].

Very recently, Yuen construct a class of self-similar solutions with elliptic symmetry for the Euler and Navier-Stokes equations with pressure:

$$\begin{cases} \rho_t + \nabla\cdot(\rho\vec{u}) = 0 \\ \rho\left[\frac{\partial}{\partial t}u_i + \vec{u}\cdot\nabla u_i\right] + K\frac{\partial}{\partial x_i}\rho^{\gamma} = \kappa_1\frac{\partial}{\partial x_i}(\nabla\cdot\vec{u}) + \kappa_2\frac{\partial}{\partial x_i}\operatorname{div} u_i \text{ for } i = 1,2,....,N \end{cases} \tag{4}$$

with constants $K > 0$ and $\gamma \geq 1$.
The explicit expression for the solutions are

$$\begin{cases} \rho = \frac{f(s)}{\prod\limits_{k=1}^{N} a_k} \\ u_i = \frac{\dot{a}_i}{a_i}(x_i + d_i) \text{ for } i = 1,2,....,N \end{cases} \tag{5}$$

where

$$f(s) = \begin{cases} \alpha e^{-\frac{\xi}{2K}s} & \text{for } \gamma = 1 \\ \max\left(\left(-\frac{\xi(\gamma-1)}{2K\gamma}s + \alpha\right)^{\frac{1}{\gamma-1}},\ 0\right) & \text{for } \gamma > 1 \end{cases} \tag{6}$$

with $s = \sum\limits_{k=1}^{N}\frac{(x_k+d_k)^2}{a_k(t)^2}$, arbitrary constants $\alpha \geq 0$, d_k and ξ;
and the auxiliary functions $a_i = a_i(t)$ satisfy the Emden dynamical system:

$$\begin{cases} \ddot{a}_i = \frac{\xi}{a_i\left(\prod\limits_{k=1}^{N} a_k\right)^{\gamma-1}}, \text{ for } i = 1,2,....,N \\ a_i(0) = a_{i0} > 0,\ \dot{a}_i(0) = a_{i1} \end{cases} \tag{7}$$

with arbitrary constants a_{i0} and a_{i1}.

In the paper, we continue to construct the corresponding results for the pressureless density-dependent Navier-Stokes equations (2) with elliptical symmetry. We successfully reduce the pressureless Navier-Stokes equations with density-dependent viscosity into $1+N$ differential functional equations in the following two theorems:

Theorem 1 *To the pressureless density-dependent Navier-Stokes equations (2) with $\kappa_1 > 0$ and $\kappa_2 = 0$ in R^N, there exists a family of solutions:*

$$\begin{cases} \rho = \frac{f(s)}{\prod\limits_{k=1}^{N} a_k} \\ u_i = \frac{\dot{a}_i}{a_i}(x_i + d_i) \text{ for } i = 1,2,....,N \end{cases} \tag{8}$$

where

$$f(s) = \begin{cases} \alpha e^{-\frac{\xi}{2\kappa_1}s} & \text{for } \theta = 1 \\ \max\left(\left(-\frac{\xi(\theta-1)}{2\kappa_1\gamma}s + \alpha\right)^{\frac{1}{\theta-1}},\ 0\right) & \text{for } \theta \neq 1 \end{cases} \tag{9}$$

with $s = \sum\limits_{k=1}^{N} \frac{(x_k+d_k)^2}{a_k(t)^2}$, *arbitrary constants* $\alpha \geq 0$, d_k *and* ξ;
and the auxiliary functions $a_i = a_i(t)$ *satisfy the Emden dynamical system:*

$$\begin{cases} \ddot{a}_i(t) = \dfrac{-\xi\left(\sum_{k=1}^{N} \frac{\dot{a}_k(t)}{a_k(t)}\right)}{a_i(t)\left(\prod\limits_{k=1}^{N} a_k(t)\right)^{\theta-1}} \text{ for } i = 1,2,...,N \\ a_i(0) = a_{i0} > 0, \ \dot{a}_i(0) = a_{i1} \end{cases} \tag{10}$$

with arbitrary constants a_{i0} *and* a_{i1}.
In particular, for $\xi < 0$,
(1) if all $a_{i1} < 0$, *the solutions (8) blow up on or before the finite time*

$$T = \min(-a_{i0}/a_{i1} : a_{1i} < 0, \ i = 1,2,...,N); \tag{11}$$

(2) if all $a_{i1} \geq 0$ *the solutions (8) exist globally.*

and

Theorem 2 *To the pressureless density-dependent Navier-Stokes equations (2) with* $\kappa_1 = 0$ *and* $\kappa_2 > 0$ *in* R^N, *there exists a family of solutions:*

$$\begin{cases} \rho = \dfrac{f(s)}{\prod\limits_{k=1}^{N} a_k} \\ u_i = \frac{\dot{a}_i}{a_i}(x_i + d_i) \text{ for } i = 1,2,....,N \end{cases} \tag{12}$$

where

$$f(s) = \begin{cases} \alpha e^{-\frac{\xi}{2\kappa_2}s} & \text{for } \theta = 1 \\ \max\left(\left(-\frac{\xi(\theta-1)}{2\kappa_2\gamma}s + \alpha\right)^{\frac{1}{\theta-1}}, \ 0\right) & \text{for } \theta \neq 1 \end{cases} \tag{13}$$

with $s = \sum\limits_{k=1}^{N} \frac{(x_k+d_k)^2}{a_k(t)^2}$, *arbitrary constants* $\alpha \geq 0$, d_k *and* ξ;
and the auxiliary functions $a_i = a_i(t)$ *satisfy the Emden dynamical system:*

$$\begin{cases} \ddot{a}_i(t) = \dfrac{-\xi \dot{a}_i(t)}{a_i^{N(\theta-1)+2}(t)} \text{ for } i = 1,2,...,N \\ a_i(0) = a_{i0} > 0, \ \dot{a}_i(0) = a_{i1}. \end{cases} \tag{14}$$

with arbitrary constants a_{i0} *and* a_{i1}.
In particular, for $\xi < 0$
(1) if some $a_{i1} < 0$, *the solutions (12) blow up on or before the finite time*

$$T = \min(-a_{i0}/a_{i1} : a_{1i} < 0, \ i = 1,2,...,N); \tag{15}$$

(2) if all $a_{i1} \geq 0$ *the solutions (12) exist globally.*

Remark 3 *For* $\xi > 0$, *the weak* C^0 *solutions (8) and (12) are the breaking waves with finite mass. For* $\xi < 0$, *the masses of the classical* C^2 *solutions (8) and (12) are infinite.*

Remark 4 *Theorem 2 completely covers the result for the pressureless Navier-Stokes equations with density-dependent viscosity of Yuen's paper [11].*

2 Reduction of the Density-Dependent System

Similar to Yuen's paper [14] for the Euler and Navier-Stokes equations (4), we apply the Yeung and Yuen' lemma in [13] and [10] to obtain the explicit expression for the mass equation:

Lemma 5 (Lemma 1 in [10]) *For the equation of conservation of mass:*

$$\rho_t + \nabla\cdot(\rho\vec{u}) = 0, \tag{16}$$

there exist solutions,

$$\begin{cases} \rho = \dfrac{f\left(\frac{x_1+d_1}{a_1(t)}, \frac{x_2+d_2}{a_2(t)}, ..., \frac{x_N+d_N}{a_N(t)}\right)}{\prod\limits_{i=1}^{N} a_i(t)} \\ u_i = \frac{\dot{a}_i(t)}{a_i(t)}(x_i+d_i) \ \textit{for } i = 1, 2, ..., N \end{cases} \tag{17}$$

with an arbitrary C^1 function $f \geq 0$ and $a_i(t) > 0$ and constants d_i.

We can reduce the density-dependent Navier-Stokes equations (2) into $1 + N$ differential functional equations:

Proof of Theorem 2. By Lemma 16, our functions (5) satisfy the mass equation $(2)_1$.

First, we define the self-similar variable:

$$s = \sum_{k=1}^{N} \frac{(x_k + d_k)^2}{a_k(t)^2}. \tag{18}$$

For the i-th momentum equation of the pressureless Navier-Stokes equations with density-dependent viscosity, $\kappa_1 > 0$ and $k_2 = 0$ (2), we obtain

$$\rho\left[\frac{\partial u_i}{\partial t} + \sum_{k=1}^{N} u_k \frac{\partial u_i}{\partial x_k}\right] - \kappa_1 \frac{\partial}{\partial x_i}\left(\rho^\theta \nabla\cdot\vec{u}\right) \tag{19}$$

$$= \rho\left[\frac{\partial}{\partial t}\left(\frac{\dot{a}_i}{a_i}(x_i+d_i)\right) + \left(\frac{\dot{a}_i}{a_i}(x_i+d_i)\right)\frac{\partial}{\partial x_i}\left(\frac{\dot{a}_i}{a_i}(x_i+d_i)\right)\right] - \kappa_1\theta\sum_{k=1}^{N}\frac{\dot{a}_k}{a_k}\rho^{\theta-1}\frac{\partial}{\partial x_i}\rho \tag{20}$$

$$= \rho\left\{\left[\left(\frac{\ddot{a}_i}{a_i} - \frac{(\dot{a}_i)^2}{(a_i)^2}\right)(x_i+d_i) + \frac{(\dot{a}_i)^2}{(a_i)^2}(x_i+d_i)\right] - \kappa_1\theta\sum_{k=1}^{N}\frac{\dot{a}_k}{a_k}\rho^{\theta-2}\frac{\partial}{\partial x_i}\frac{f(s)}{\prod\limits_{k=1}^{N} a_k}\right\} \tag{21}$$

$$= \rho\left\{\frac{\ddot{a}_i}{a_i}(x_i+d_i) - 2\kappa_1\theta\sum_{k=1}^{N}\frac{\dot{a}_k}{a_k}\frac{f(s)^{\theta-2}}{\left(\prod\limits_{k=1}^{N} a_k\right)^{\theta-2}}\frac{\dot{f}(s)}{\left(\prod\limits_{k=1}^{N} a_k\right)}\left(\frac{x_i+d_i}{a_i^2}\right)\right\} \tag{22}$$

$$= \frac{(x_i+d_i)\left(\sum_{k=1}^{N}\frac{\dot{a}_k}{a_k}\right)\rho}{a_i^2}\left\{\frac{\ddot{a}_i a_i}{\left(\sum_{k=1}^{N}\frac{\dot{a}_k}{a_k}\right)} - 2\kappa_1\theta\frac{f(s)^{\theta-2}\dot{f}(s)}{\left(\prod\limits_{k=1}^{N} a_k\right)^{\theta-1}}\right\} \tag{23}$$

$$= \frac{-(x_i+d_i)\left(\sum_{k=1}^{N}\frac{\dot{a}_k}{a_k}\right)\rho}{a_i^2\left(\prod\limits_{k=1}^{N} a_k\right)^{\theta-1}}\left\{\xi + 2\kappa_1\theta f(s)^{\theta-2}\dot{f}(s)\right\} \tag{24}$$

with the N-dimensional Emden dynamical system with force-force interaction:

$$\begin{cases} \ddot{a}_i(t) = \dfrac{-\xi\left(\sum_{k=1}^{N}\frac{\dot{a}_k(t)}{a_k(t)}\right)}{a_i(t)\left(\prod\limits_{k=1}^{N} a_k(t)\right)^{\theta-1}} \text{ for } i = 1, 2, ..., N \\ a_i(0) = a_{i0} > 0, \ \dot{a}_i(0) = a_{i1} \end{cases} \tag{25}$$

with arbitrary constants ξ, a_{i0} and a_{i1}.
Then, we ask the first order ordinary differential equation:

$$\begin{cases} \frac{\xi}{2\kappa_1\theta} + f(s)^{\theta-2}\dot{f}(s) = 0 \\ f(0) = \alpha \geq 0, \end{cases} \text{ or } \rho = 0. \tag{26}$$

The equation (26) can be solved by

$$f(s) = \begin{cases} \alpha e^{-\frac{\xi}{2\kappa_1\theta}s} & \text{for } \theta = 1 \\ \max\left(\left(-\frac{\xi(\theta-1)}{2\kappa_1\gamma}s + \alpha\right)^{\frac{1}{\theta-1}},\ 0\right) & \text{for } \theta \neq 1. \end{cases} \tag{27}$$

Therefore, the functions (8) are the solutions for the pressureless density-dependent Navier-Stokes equations (2).

In particular, for $\xi < 0$,

(1) if all $a_{i1} \leq 0$, we have

$$\frac{\ddot{a}_i(t)}{\left(\sum_{k=1}^N \frac{\dot{a}_k(t)}{a_k(t)}\right)} > 0 \text{ for } i = 1, 2, ..., N. \tag{28}$$

It is clear to show that the solutions (8) blow up on or before the finite time

$$T = \min(-a_{i0}/a_{i1} : a_{i1} < 0,\ i = 1, 2, ..., N); \tag{29}$$

(2) all $a_{i1} \geq 0$, similarly to (1), we have the solutions (8) exist globally.
The proof is completed. ■

The proof for Theorem 2 is similar.

Proof of Theorem 2. For the mass equation, it is the same as the proof in Theorem 1.

For the i-th momentum equation $(2)_2$, we have for $\kappa_1 = 0$ and $\kappa_2 > 0$:

$$\rho\left[\frac{\partial u_i}{\partial t} + \sum_{k=1}^N u_k \frac{\partial u_i}{\partial x_k}\right] - \kappa_2 \nabla \cdot \left(\rho^\theta \nabla u_i\right) \tag{30}$$

$$= \rho\left[\frac{\partial}{\partial t}\left(\frac{\dot{a}_i}{a_i}(x_i + d_i)\right) + \left(\frac{\dot{a}_i}{a_i}(x_i + d_i)\right)\frac{\partial}{\partial x_i}\left(\frac{\dot{a}_i}{a_i}(x_i + d_i)\right)\right] - \frac{\kappa_2\theta\dot{a}_i}{a_i}\rho^{\theta-1}\frac{\partial}{\partial x_i}\rho \tag{31}$$

$$= \rho\left\{\left[\left(\frac{\ddot{a}_i}{a_i} - \frac{(\dot{a}_i)^2}{(a_i)^2}\right)(x_i + d_i) + \frac{(\dot{a}_i)^2}{(a_i)^2}(x_i + d_i)\right] - \frac{\kappa_2\theta\dot{a}_i}{a_i}\rho^{\theta-2}\frac{\partial}{\partial x_i}\frac{f(s)}{\prod\limits_{k=1}^N a_k}\right\} \tag{32}$$

$$= \frac{(x_i + d_i)\,\dot{a}_i\rho}{a_i^2 a_i}\left\{\frac{\ddot{a}_i a_i^2}{\dot{a}_i} - 2\kappa_2\theta\frac{f(s)^{\theta-2}\dot{f}(s)}{\left(\prod\limits_{k=1}^N a_k\right)^{\theta-1}}\right\} \tag{33}$$

$$= \frac{-(x_i + d_i)\,\dot{a}_i\rho}{a_i^2 a_i\left(\prod\limits_{k=1}^N a_k\right)^{\gamma-1}}\left\{\xi + 2\kappa_2\theta f(s)^{\theta-2}\dot{f}(s)\right\} \tag{34}$$

with the N-dimensional Emden dynamical system:

$$\begin{cases} \ddot{a}_i(t) = \frac{-\xi\dot{a}_i(t)}{a_i^2(t)\left(\prod\limits_{k=1}^N a_k(t)\right)^{\gamma-1}} \text{ for } i = 1, 2, ..., N \\ a_i(0) = a_{i0} > 0,\ \dot{a}_i(0) = a_{i1} \end{cases} \tag{35}$$

with arbitrary constants ξ, a_{i0} and a_{i1}.
Similarly, we require:

$$\begin{cases} \frac{\xi}{2\kappa_2\theta}+f(s)^{\theta-2}\dot{f}(s)=0 \\ f(0)=\alpha\geq 0, \end{cases} \text{ or } \rho=0. \tag{36}$$

That is

$$f(s)=\begin{cases} \alpha e^{-\frac{\xi}{2\kappa_2\theta}s} & \text{for } \theta=1 \\ \max\left(\left(-\frac{\xi(\theta-1)}{2\kappa_2\gamma}s+\alpha\right)^{\frac{1}{\theta-1}},\ 0\right) & \text{for } \theta\neq 1. \end{cases} \tag{37}$$

Thus, we verified that the functions (12) are the solutions for the Navier-Stokes equations (2). It is clear to see the following properties of the constructed solutions.

In particular, for $\xi<0$,
(1) if some $a_{i1}\leq 0$, the solutions (12) blow up on or before the finite time

$$T=\min(-a_{i0}/a_{i1}:a_{i1}<0,\ i=1,2,...,N); \tag{38}$$

(2) if all $a_{i1}\geq 0$, that the solutions (12) exist globally.
The proof is completed. ■

3 Conclusion and Discussion

In this article, we successfully reduce the density-dependent Navier-Stokes equations (2) into the $1+N$ differential functional equations:

$$(f(s),a_i(t) \text{ for } i=1,2,....N), \tag{39}$$

to obtain the self-similar solutions with elliptic symmetry.
Here, the velocity $\vec{u}$ is constructed by the novel Emden dynamical systems with force-force interaction (10):

$$\begin{cases} \ddot{a}_i(t)=\dfrac{-\xi\left(\sum_{k=1}^{N}\frac{\dot{a}_k(t)}{a_k(t)}\right)}{a_i(t)\left(\prod\limits_{k=1}^{N}a_k(t)\right)^{\theta-1}} \text{ for } i=1,2,...,N \\ a_i(0)=a_{i0}>0,\ \dot{a}_i(0)=a_{i1} \end{cases} \tag{40}$$

or (14)

$$\begin{cases} \ddot{a}_i(t)=\frac{-\xi\dot{a}_i(t)}{a_i^{N(\theta-1)+2}(t)} \text{ for } i=1,2,...,N \\ a_i(0)=a_{i0}>0,\ \dot{a}_i(0)=a_{i1}. \end{cases} \tag{41}$$

Comparing with the Emden dynamical system (7):

$$\begin{cases} \ddot{a}_i=\dfrac{\xi}{a_i\left(\prod\limits_{k=1}^{N}a_k\right)^{\gamma-1}}, \text{ for } i=1,2,....,N \\ a_i(0)=a_{i0}>0,\ \dot{a}_i(0)=a_{i1} \end{cases} \tag{42}$$

for the compressible Euler and Navier-Stokes equations without density-dependent viscosity (4), the corresponding ones for the pressureless density-dependent Navier-Stokes equations (2) are much complicated. It is very interesting to investigate this kind of Emden dynamical systems (40) and (41) firstly by numerical methods, because it is seen to be difficult to determine the all blowup sets, blowup times and asymptotic analysis of the solutions by rigorous mathematics. Then, it is natural to conjecture that there exist limit cycles or chaos (especially for the finite mass cases with $\xi>0$) for the Emden dynamical systems (40) and (41) with $N\geq 2$. Thus, further researches on proving or disproving the conjecture are expected to better understand the constructed flows (8) and (12) for the system with density-dependent viscosity (2).

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